Frequency dependence of the photonic noise spectrum in an absorbing or amplifying diffusive medium

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Abstract

A theory is presented for the frequency dependence of the power spectrum of photon current fluctuations originating from a disordered medium. Both the cases of an absorbing medium ("grey body") and of an amplifying medium ("random laser") are considered in a waveguide geometry. The semiclassical approach (based on a Boltzmann-Langevin equation) is shown to be in complete agreement with a fully quantum mechanical theory, provided that the effects of wave localization can be neglected. The width of the peak in the power spectrum around zero frequency is much smaller than the inverse coherence time, characteristic for black-body radiation. Simple expressions for the shape of this peak are obtained, in the absorbing case, for waveguide lengths large compared to the absorption length, and, in the amplifying case, close to the laser threshold.

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I. INTRODUCTION

The noise power spectrum of a black body is frequency independent for frequencies below the absorption band width. The inverse of the band width is the coherence time $\tau_{\rm coh}$ of the radiation [1], which for a black body is the longest relevant time scale — hence the white noise spectrum $P(\Omega)$ for $\Omega \lesssim 1/\tau_{\rm coh}$. In a weakly absorbing, strongly scattering medium there appear two longer time scales: The absorption time τ_a and the time L^2/D it takes to diffuse (with diffusion constant D) through the medium (of length L). As a consequence, $P(\Omega)$ for such a weakly-absorbing medium (sometimes called a "grey body") starts to decay at much lower frequencies than for a black body having the same coherence time.

Although there is by now a substantial literature on the theory of grey-body radiation [2–7], the results have been limited to either the zero or high-frequency limits of the noise spectrum (or, equivalently, to short or long photodetection times). In the present work we remove this limitation, by computing $P(\Omega)$ for a diffusive medium for arbitrary ratios of Ω, $1/\tau_a$, and D/L^2. We compare two different approaches in a waveguide geometry: One which is fully quantum mechanical (based on random-matrix theory [7,8]) and another which is semiclassical (based on a Boltzmann-Langevin equation [9]). Each method has

its advantages and disadvantages: The quantum theory includes interference effects, which are ignored in the semiclassical theory, but it is mathematically more involved. Complete agreement between the two approaches is obtained in the limit that the waveguide length L is much smaller than the localization length (equal to the mean free path times the number of propagating modes).

The results for absorbing media can be applied directly to linear amplifiers, by formally changing the sign of the temperature and the absorption time. Loudon and coworkers [10,11] used this relationship to calculate the noise power spectrum of a waveguide without disorder. The generalization to a diffusive medium presented here describes a random laser [12] below threshold.

The outline of this paper is as follows. We start with the semiclassical approach, presenting a general solution of the Boltzmann-Langevin equation in Sec. II and applying it to a waveguide geometry in Sec. III. The quantum mechanical approach is developed in Sec. IV. For the quantum theory we need the correlator of reflection and transmission matrices at different frequencies. These are calculated in the appendix, using the random-matrix method of Ref. [13]. We discuss our findings in Sec. V.

II. SEMICLASSICAL THEORY

Starting point of the semiclassical theory is the Boltzmann-Langevin equation for photons of Ref. [9]. We first consider an absorbing medium (in equilibrium at temperature T), leaving the amplifying case for the end of this section. We make the diffusion approximation, valid if the mean free path l is the shortest length scale in the system (but still large compared to the wavelength). The fluctuating number density $n(\omega, \mathbf{r}, t)$ and current density $\mathbf{j}(\omega, \mathbf{r}, t)$ of photons at frequency ω, position $\mathbf{r}$, and time t are related by [9]

$$\mathbf{j} = -D\frac{\partial n}{\partial \mathbf{r}} + \boldsymbol{\mathcal{L}}_1, \tag{2.1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{j} = D\xi_a^{-2}(\rho f - n) + \mathcal{L}_0. \tag{2.2}$$

Here $D = \frac{1}{3}cl$ is the diffusion constant, $\xi_a = \sqrt{D\tau_a}$ is the absorption length (with τ_a the absorption time), $\rho = 4\pi\omega^2(2\pi c)^{-3}$ is the density of states (not counting polarizations), and $f = [\exp(\hbar\omega/kT) - 1]^{-1}$ is the Bose-Einstein function. We assume $\xi_a \gg l$. The fluctuating source terms $\mathcal{L}_0$ and $\boldsymbol{\mathcal{L}}_1$ have zero mean and correlators

$$\overline{\mathcal{L}_0(\omega, \mathbf{r}, t)\mathcal{L}_0(\omega', \mathbf{r}', t')} = \delta(\omega - \omega')\delta(t - t')\delta(\mathbf{r} - \mathbf{r}')D\xi_a^{-2}(2f\bar{n} + \rho f + \bar{n}), \tag{2.3a}$$

$$\overline{\mathcal{L}_{1\alpha}(\omega, \mathbf{r}, t)\mathcal{L}_{1\beta}(\omega', \mathbf{r}', t')} = 2\delta_{\alpha\beta}\delta(\omega - \omega')\delta(t - t')\delta(\mathbf{r} - \mathbf{r}')D\bar{n}(1 + \bar{n}/\rho). \tag{2.3b}$$

The cross-correlator of $\mathcal{L}_0$ and $\boldsymbol{\mathcal{L}}_1$ is given in Ref. [9], but will not be needed. Combining Eqs. (2.1) and (2.2) we find equations for the mean $\bar{n}$ and the fluctuations δn of the photon number density $n = \bar{n} + \delta n$,

$$-\frac{1}{D}\frac{\partial \bar{n}}{\partial t} + \frac{\partial^2 \bar{n}}{\partial \mathbf{r}^2} - \frac{\bar{n}}{\xi_a^2} = -\frac{\rho f}{\xi_a^2}, \tag{2.4}$$

$$-\frac{1}{D}\frac{\partial \delta n}{\partial t} + \frac{\partial^2 \delta n}{\partial \mathbf{r}^2} - \frac{\delta n}{\xi_a^2} = \frac{1}{D}\frac{\partial}{\partial \mathbf{r}} \cdot \boldsymbol{\mathcal{L}}_1 - \frac{\mathcal{L}_0}{D}. \tag{2.5}$$

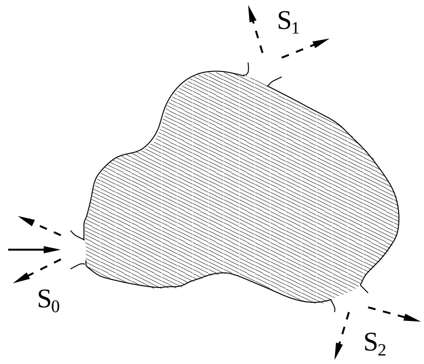

FIG. 1. Thermal radiation (solid arrow) is incident through port S_0 on an absorbing disordered medium (shaded). The outgoing radiation (dashed arrows) is absorbed by photodetectors.

We present a general solution for the multiport geometry of Fig. 1. Thermal radiation is incident through the port S_0 and can leave the system via ports S_0, S_1, S_2, …, where it is absorbed by photodetectors. The corresponding boundary conditions are $n(\omega, \mathbf{r}, t)|_{\mathbf{r}\in S_p} = n_{\rm in}(\omega, t)\delta_{p0}$. We assume that the closed boundaries Σ of the system (with volume V) are perfectly reflecting. The separation of the ports is of order $L \gg l$. In what follows we assume detection of outgoing radiation in a narrow frequency interval $\delta\omega$ around ω. We require that $\delta\omega$ is small both compared to ω and to $1/\tau_{\rm coh}$. To minimize the notations in this section we omit the frequency argument ω and use units in which $\delta\omega \equiv 1$. (We will reinsert $\delta\omega$ in the next section.)

The Green function of the differential equations (2.4) and (2.5) in the Fourier representation with respect to the time argument satisfies

$$\left(\frac{\partial^2}{\partial \mathbf{r}^2} - \xi_a^{-2} + \frac{i\Omega}{D}\right) G(\mathbf{r}, \mathbf{r}', \Omega) = \delta(\mathbf{r} - \mathbf{r}'). \tag{2.6}$$

[Fourier transforms are defined as $f(\Omega) = \int_{-\infty}^{\infty} dt\, e^{i\Omega t} f(t)$.] For frequency resolved detection we require $\Omega \ll \delta\omega$. We impose the boundary conditions

$$G(\mathbf{r}, \mathbf{r}', \Omega)|_{\mathbf{r}\in S_p} = 0, \;\; p = 0, 1, 2, \ldots, \tag{2.7a}$$

$$\mathbf{\Sigma} \cdot \frac{\partial G(\mathbf{r}, \mathbf{r}', \Omega)}{\partial \mathbf{r}}|_{\mathbf{r}\in\Sigma} = 0, \tag{2.7b}$$

where $\mathbf{\Sigma}$ denotes the outward normal direction to the surface Σ. We consider separately the mean and the fluctuations of the photon number and current densities.

A. Mean solution

The average photon density satisfying Eq. (2.4) can be expressed in Fourier representation in terms of the Green function (2.6),

$$\bar{n}(\mathbf{r},\Omega) = -2\pi\rho f \xi_a^{-2}\delta(\Omega)\int\limits_V d\mathbf{r}'\; G(\mathbf{r},\mathbf{r}',0) + \bar{n}_{\rm in}(\Omega)\int\limits_{S_0} d\mathbf{S}'\cdot\frac{\partial G(\mathbf{r},\mathbf{r}',\Omega)}{\partial\mathbf{r}'}. \tag{2.8}$$

Substituting this formula into the expression for the current (2.1) and integrating over the area S_p one obtains the mean outgoing current $\bar{I}_p$ through port $p \neq 0$,

$$\begin{aligned}\bar{I}_p(\Omega) = 2\pi\rho D f\xi_a^{-2}\delta(\Omega)\int\limits_{S_p} d\mathbf{S}\cdot\int\limits_V d\mathbf{r}'\;\frac{\partial G(\mathbf{r},\mathbf{r}',0)}{\partial\mathbf{r}}\\ - D\bar{n}_{\rm in}(\Omega)\int\limits_{S_p} dS_\alpha\int\limits_{S_0} dS'_\beta\;\frac{\partial^2 G(\mathbf{r},\mathbf{r}',\Omega)}{\partial r_\alpha\partial r'_\beta}.\end{aligned} \tag{2.9}$$

(Summation over the repeating Greek indices is implied.) The first term $\propto \delta(\Omega)$ is the time-independent mean thermal radiation from the medium. The second term is that part of the mean radiation entering through port 0 that leaves the medium through one of the other ports. (The restriction to $p \neq 0$ is not essential but simplifies the general formulas considerably, so we will make this restriction in what follows.)

B. Fluctuations

The fluctuations in the number density follow in a similar way from the Green function and Eq. (2.5),

$$\delta n(\mathbf{r},\Omega) = \frac{1}{D}\int\limits_V d\mathbf{r}'\; G(\mathbf{r},\mathbf{r}',\Omega)\left(\frac{\partial}{\partial\mathbf{r}'}\cdot\boldsymbol{\mathcal{L}}_1(\mathbf{r}',\Omega) - \mathcal{L}_0(\mathbf{r}',\Omega)\right) + \delta n_{\rm in}(\Omega)\int\limits_{S_0} d\mathbf{S}'\cdot\frac{\partial G(\mathbf{r},\mathbf{r}',\Omega)}{\partial\mathbf{r}'}. \tag{2.10}$$

The fluctuation of the current density is then given by Eq. (2.1),

$$\begin{aligned}\delta j_\alpha(\mathbf{r},\Omega) = \int\limits_V d\mathbf{r}'\left(G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\Omega)\mathcal{L}_{1\beta}(\mathbf{r}',\Omega) + \frac{\partial G(\mathbf{r},\mathbf{r}',\Omega)}{\partial r_\alpha}\mathcal{L}_0(\mathbf{r}',\Omega)\right)\\ - D\delta n_{\rm in}(\Omega)\int\limits_{S_0} dS'_\beta\; G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\Omega).\end{aligned} \tag{2.11}$$

We have defined

$$G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\Omega) = \frac{\partial^2 G(\mathbf{r},\mathbf{r}',\Omega)}{\partial r_\alpha\partial r'_\beta} + \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \tag{2.12}$$

We seek the correlator of the current fluctuations

$$C_{\alpha\beta}(\mathbf{r},\Omega;\mathbf{r}',\Omega') = \overline{\delta j_\alpha(\mathbf{r},\Omega)\delta j_\beta(\mathbf{r}',\Omega)} \tag{2.13}$$

for $\mathbf{r} \in S_p$, $\mathbf{r}' \in S_q$ with $p, q \neq 0$. With the help of Eqs. (2.3) and (2.11) it can be expressed as

$$C_{\alpha\beta}(\mathbf{r},\Omega;\mathbf{r}',\Omega') = \frac{D}{\xi_a^2}\int\limits_V d\mathbf{r}''\frac{\partial G(\mathbf{r},\mathbf{r}'',\Omega)}{\partial r_\alpha}\frac{\partial G(\mathbf{r}',\mathbf{r}'',\Omega')}{\partial r'_\beta}[(2f+1)\bar{n}(\mathbf{r}'',\Omega+\Omega')+\rho f]$$
$$+2D\int\limits_V d\mathbf{r}'' G_{\alpha\gamma}(\mathbf{r},\mathbf{r}'',\Omega)G_{\beta\gamma}(\mathbf{r}',\mathbf{r}'',\Omega')\Big[\bar{n}(\mathbf{r}'',\Omega+\Omega')+\frac{1}{\rho}\int\frac{d\Omega''}{2\pi}\bar{n}(\mathbf{r}'',\Omega+\Omega'')\bar{n}(\mathbf{r}'',\Omega'-\Omega'')\Big]. \tag{2.14}$$

Following Ref. [9], we have neglected the term $\propto \delta n_{\rm in}$ in Eq. (2.11) (smaller by a factor l/L) and the cross-correlator $\overline{\mathcal{L}_0\boldsymbol{\mathcal{L}}_1}$ (smaller by a factor l/ξ_a).

We now integrate $\mathbf{r}$ and $\mathbf{r}'$ over S_p and S_q to obtain the correlator of the total currents through ports p and q,

$$C_{pq}(\Omega,\Omega') = \int\limits_{S_p} dS_\alpha \int\limits_{S_q} dS'_\beta\; C_{\alpha\beta}(\mathbf{r},\Omega;\mathbf{r}',\Omega') = C^{(1)}_{pq}(\Omega,\Omega') + C^{(2)}_{pq}(\Omega,\Omega'). \tag{2.15}$$

The first term $C^{(1)}_{pq}$ contains the contribution from the terms linear in the number density $\bar{n}$ in Eq. (2.14). Performing integration by parts and using Eqs. (2.6)–(2.8) we find that this term vanishes for $p \neq q$. For $p = q$ it contains the mean current,

$$C^{(1)}_{pq}(\Omega,\Omega') = \delta_{pq}\bar{I}_p(\Omega+\Omega'). \tag{2.16}$$

For a time-independent mean current $\bar{I}_p$ one has a white-noise spectrum $C^{(1)}_{pq}(\Omega,\Omega') = 2\pi\delta_{pq}\delta(\Omega+\Omega')\bar{I}_p$. This is the usual shot noise, corresponding to Poissonian statistics of the current fluctuations. The second term $C^{(2)}_{pq}$ describes the deviations from Poissonian statistics. It arises from terms in Eq. (2.14) that are quadratic in $\bar{n}$. Performing again an integration by parts, one finds

$$C^{(2)}_{pq}(\Omega,\Omega') = \frac{2D}{\rho}\int\limits_{S_p} dS_\alpha \int\limits_{S_q} dS'_\beta \int\limits_V d\mathbf{r}'' \int \frac{d\Omega''}{2\pi}\frac{\partial \bar{n}(\mathbf{r}'',\Omega+\Omega'')}{\partial r''_\gamma}\frac{\partial \bar{n}(\mathbf{r}'',\Omega'-\Omega'')}{\partial r''_\gamma} \times \frac{\partial G(\mathbf{r},\mathbf{r}'',\Omega)}{\partial r_\alpha}\frac{\partial G(\mathbf{r}',\mathbf{r}'',\Omega')}{\partial r'_\beta}. \tag{2.17}$$

Equation (2.17) together with Eq. (2.8) is the result that we need for our analysis of the frequency dependence of the noise spectrum.

C. Amplifying medium

The extension of our general formulas to an amplifying medium (in the linear regime below the laser threshold) is straightforward [9]: We assume that the frequency ω at which we are detecting the radiation is close to the frequency of an atomic transition with (on average) $N_{\rm upper}$ and $N_{\rm lower}$ atoms in the upper and lower state. Then the Bose-Einstein function can be replaced by the population inversion factor $f = N_{\rm upper}(N_{\rm lower} - N_{\rm upper})^{-1}$. This factor is negative in the amplifying case (when $N_{\rm upper} > N_{\rm lower}$), with $f = -1$ for a complete population inversion. (Equivalently, one can evaluate f at a negative temperature [11], with $T \to 0^-$ for complete inversion.) An amplifying medium has a negative absorption time $\tau_a = \xi_a^2/D$. We can account for this by taking ξ_a imaginary. With these two substitutions for f and ξ_a our formulas for an absorbing medium carry over to the amplifying case.

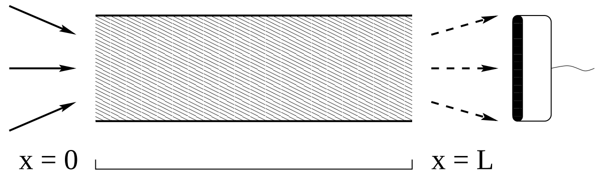

FIG. 2. Thermal radiation (solid arrows) is incident on a waveguide containing an absorbing or amplifying disordered medium. The transmitted radiation (dashed arrows) is absorbed by a photodetector.

III. WAVEGUIDE GEOMETRY

For the application of our general formulas we consider a waveguide geometry (see Fig. 2). The waveguide has length L and cross-sectional area A, corresponding to $N = \omega^2 A/4\pi c^2$ propagating modes (not counting polarizations) at frequency ω. We abbreviate $s = L/\xi_a$. We consider a stationary incident current $I_0 = \frac{1}{4}cA\delta\omega\bar{n}_{\rm in} = (N\delta\omega/2\pi\rho)\bar{n}_{\rm in}$, and calculate the noise power spectrum of the transmitted current,

$$P(\Omega) = \int_{-\infty}^{\infty} dt\; e^{i\Omega t}\; \overline{\delta I(t)\delta I(0)}. \tag{3.1}$$

In terms of the correlator of the previous section, one has $C_{11}(\Omega,\Omega') = 2\pi P(\Omega)\delta(\Omega+\Omega')$.

A. Absorbing medium

We calculate the noise power from Eqs. (2.8) and (2.17), using the Green function

$$G(x,x',\Omega) = -\xi_a \frac{\sinh\left[(x_</\xi_a)\sqrt{1-i\Omega\tau_a}\right]\sinh\left[(s-x_>/\xi_a)\sqrt{1-i\Omega\tau_a}\right]}{\sinh\left[s\sqrt{1-i\Omega\tau_a}\right]}, \tag{3.2}$$

where $x_<$ and $x_>$ are the smallest and largest of x, x', respectively. The mean photon density is time independent. In Fourier representation one has, from Eq. (2.8),

$$\begin{aligned}\bar{n}(x,\Omega) = 2\pi\delta(\Omega)\frac{\rho f}{\sinh s}\Big(\sinh s - \sinh(x/\xi_a) - \sinh(s - x/\xi_a)\Big)\\ + 2\pi\delta(\Omega)\bar{n}_{\rm in}\frac{\sinh(s-x/\xi_a)}{\sinh s}.\end{aligned} \tag{3.3}$$

The mean current $\bar{I} = \bar{I}_{\rm th} + \bar{I}_{\rm trans}$ is the sum of the thermal radiation from the medium

$$\bar{I}_{\rm th} = \frac{4Df}{c\xi_a}(N\delta\omega/2\pi)\tanh(s/2) \tag{3.4}$$

and the transmitted incident current

$$\bar{I}_{\rm trans} = \frac{4DI_0}{c\xi_a \sinh s}. \tag{3.5}$$

Substitution of Eqs. (3.2) and (3.3) into Eq. (2.17) yields the super-Poissonian noise $P-\bar{I}$ as a sum of three terms, $P-\bar{I}=P_{\rm th}+P_{\rm trans}+P_{\rm ex}$, with

$$P_{\rm th}(\Omega)=\frac{8Df^2}{c\xi_a}(N\delta\omega/2\pi)\int_0^s ds'\,\left(\frac{\cosh(s-s')-\cosh s'}{\sinh\; s}\right)^2 K(s',s), \tag{3.6}$$

$$P_{\rm trans}(\Omega)=\frac{8DI_0^2}{c\xi_a}(2\pi/N\delta\omega)\int_0^s ds'\,\frac{\cosh^2(s-s')}{\sinh^2\; s}K(s',s), \tag{3.7}$$

$$P_{\rm ex}(\Omega)=\frac{16DfI_0}{c\xi_a}\int_0^s ds'\,\frac{[\cosh s'-\cosh(s-s')]\cosh(s-s')}{\sinh^2\; s}K(s',s). \tag{3.8}$$

We have defined

$$K(s',s)=\left|\frac{\sinh(s'\sqrt{1-i\Omega\tau_a})}{\sinh(s\sqrt{1-i\Omega\tau_a})}\right|^2. \tag{3.9}$$

The two terms $P_{\rm trans}$ and $P_{\rm th}$ describe separately the noise power of the transmitted incident current and of the thermal current from the medium. The term $P_{\rm ex}$ is the excess noise due to the beating of the incident radiation with the thermal fluctuations from the medium.

The three contributions are plotted separately in Fig. 3. For $L\gg\xi_a$ the frequency dependence simplifies to

$$P_{\rm th}(\Omega)=\frac{f\bar{I}_{\rm th}}{1+x}, \tag{3.10}$$

$$P_{\rm trans}(\Omega)=\frac{c\xi_a\bar{I}_{\rm trans}^2}{16D}(2\pi/N\delta\omega)\left(\frac{1-e^{-2s(x-1)}}{x-1}+\frac{3x+2}{x^2+x}\right), \tag{3.11}$$

$$P_{\rm ex}(\Omega)=f\bar{I}_{\rm trans}\frac{1+2x}{x+x^2}, \tag{3.12}$$

where we have defined

$$x=\mathrm{Re}\,\sqrt{1-i\Omega\tau_a}=[\tfrac{1}{2}(1+\Omega^2\tau_a^2)^{1/2}+\tfrac{1}{2}]^{1/2}. \tag{3.13}$$

As discussed in Ref. [9] (for the zero-frequency case) the result for $P_{\rm trans}$ requires that the incident radiation is in a thermal state, at some temperature T_0. (The quantity $f(\omega,T_0)=I_0(2\pi/N\delta\omega)$ is the corresponding value of the Bose-Einstein function.) There is no such requirement for $P_{\rm th}$ and $P_{\rm ex}$, which are independent of the incident state. For $T_0\gg T$ we may generally neglect $P_{\rm th}$ and $P_{\rm ex}$ relative to $P_{\rm trans}$, so that $P=\bar{I}_{\rm trans}+P_{\rm trans}$. However, if the incident radiation is in a coherent state, then $P_{\rm trans}\equiv 0$ and since for sufficiently large I_0 we may neglect $P_{\rm th}$, we have in this case $P=\bar{I}_{\rm trans}+P_{\rm ex}$. The contribution $P_{\rm th}$ is important mainly in the absence of external illumination, when $P=\bar{I}_{\rm th}+P_{\rm th}$.

B. Amplifying medium

The results for an amplifying medium are obtained by the substitution $\xi_a\to i\xi_a$, $f\to N_{\rm upper}(N_{\rm lower}-N_{\rm upper})^{-1}$, cf. Sec. IIC. The frequency dependence of $P_{\rm th}, P_{\rm trans}$, and $P_{\rm ex}$ following from Eqs. (3.6)–(3.8) is plotted in Fig. 4 for lengths L below the laser threshold at $L=\pi\xi_a$.

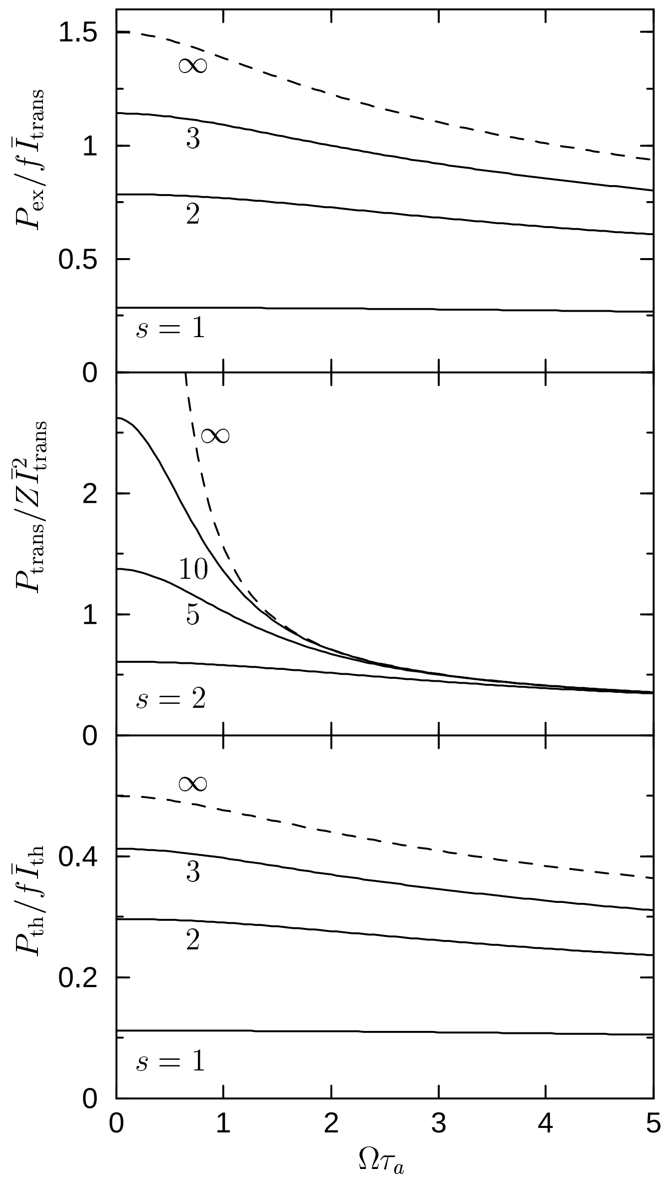

FIG. 3. Frequency dependence of the three super-Poissonian contributions to the noise power, $P - \bar{I} = P_{\rm th} + P_{\rm trans} + P_{\rm ex}$, for different values of $s = L/\xi_a$ in an absorbing waveguide. The solid curves are computed from Eqs. (3.6)–(3.8), the dashed curves are the large-s asymptotes (3.10)–(3.12). The parameter Z is defined as $Z = (c\xi_a/2D)(2\pi/N\delta\omega)$.

C. Cross-correlator

In the absence of any incident radiation, the noise $P = \bar{I}_{\rm th} + P_{\rm th}$ is due entirely to the thermal fluctuations in the medium. The current fluctuations at the two ends of the waveguide are correlated, as measured by the cross-correlator

$$P_{12}(\Omega) = \int_{-\infty}^{\infty} dt\ e^{i\Omega t}\ \overline{\delta I_1(t)\delta I_2(0)}. \tag{3.14}$$

From Eqs. (2.17), (3.2), and (3.3) we obtain

$$P_{12}(\Omega) = \frac{8Df^2}{c\xi_a}(N\delta\omega/2\pi)\int_0^s ds'\ \left(\frac{\cosh{(s-s')} - \cosh s'}{\sinh\ s}\right)^2$$

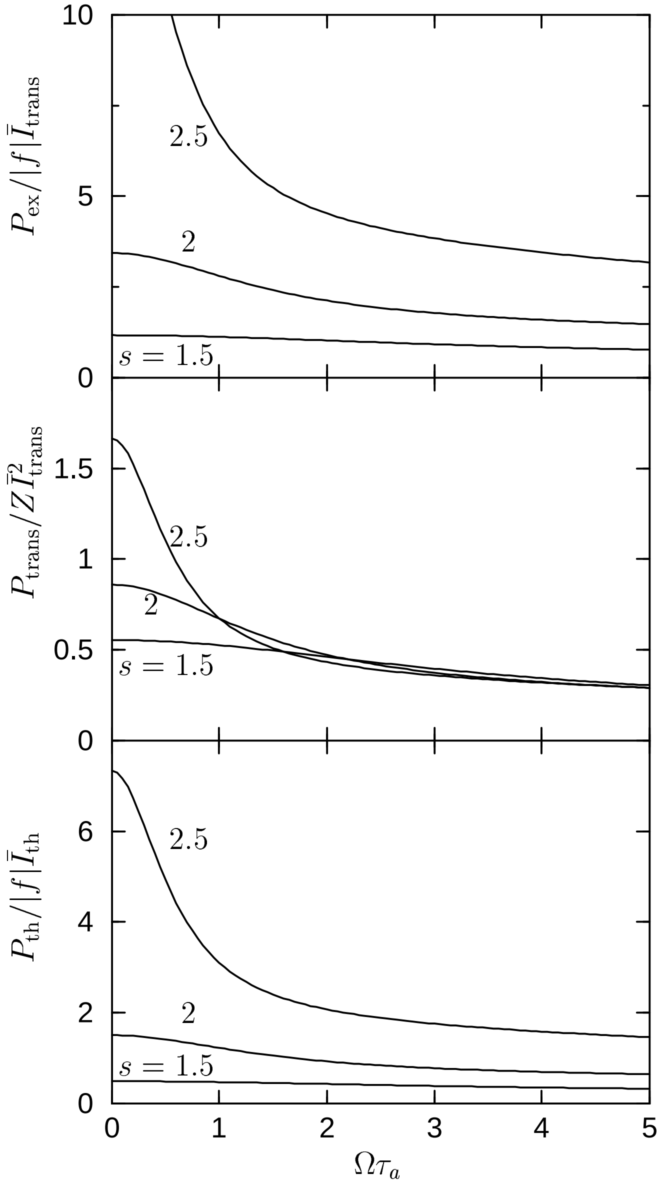

FIG. 4. Same as Fig. 3, for the case of an amplifying waveguide. The laser threshold occurs at $s = \pi$.

$$\times \frac{\sinh[s'\sqrt{1-i\Omega\tau_a}]\sinh[(s-s')\sqrt{1+i\Omega\tau_a}]}{|\sinh[s\sqrt{1-i\Omega\tau_a}]|^2}. \tag{3.15}$$

The cross-correlator is plotted in Fig. 5 for both the absorbing and amplifying cases. The outgoing currents at the two ends of the waveguide are anti-correlated for $\Omega\tau_a \gg 1$.

IV. COMPARISON WITH QUANTUM THEORY

A fully quantum mechanical theory for the photocount distribution of a disordered medium was developed in Refs. [7,8]. In this section we verify that it agrees with the semiclassical results of the previous section. We consider the same system of Fig. 2, a disordered waveguide with a photodetector at one end and a stationary current incident at the other end. We assume that the incident current originates from a thermal source at temperature T_0. The photocount distribution is the distribution of the number of photons $n(t)$

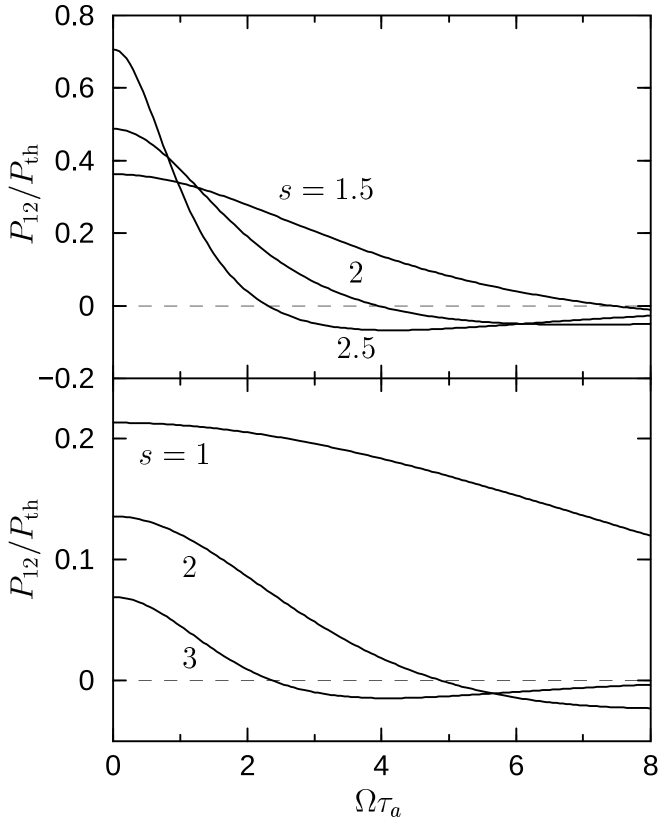

FIG. 5. Frequency dependence of the cross-correlator of the outgoing current at the two ends of the waveguide, in the absence of any external illumination. Computed from Eq. (3.15) for the absorbing case (lower panel) and amplifying case (upper panel).

counted (with unit quantum efficiency) in the time interval $(0, t)$. Substitution of $I = dn/dt$ in the definition (3.1) of the noise power $P(\Omega)$ leads to a relation with the variance $\mathrm{Var}\, n(t)$ of the photocount,

$$P(\Omega) = -\Omega^2 \int_0^\infty dt\, \mathrm{Var}\, n(t) \cos \Omega t, \tag{4.1a}$$

$$\mathrm{Var}\ n(t) = -\frac{2}{\pi} \int_0^\infty d\Omega\ \Omega^{-2} P(\Omega) \left(\cos \Omega t - 1\right). \tag{4.1b}$$

The variance can be separated into two terms, $\mathrm{Var}\, n(t) = \bar{n}(t) + \kappa(t) = t\bar{I} + \kappa(t)$, with $\kappa(t)$ the second factorial cumulant. The term $t\bar{I}$, substituted into Eq. (4.1a), gives the frequency-independent shot noise contribution $\bar{I}$ to the power spectrum,

$$P(\Omega) = \bar{I} - \Omega^2 \int_0^\infty dt\, \kappa(t) \cos \Omega t. \tag{4.2}$$

The cumulant $\kappa = \kappa_{\rm trans} + \kappa_{\rm th} + \kappa_{\rm ex}$ contains separate contributions from the transmitted incident radiation and thermal fluctuations in the medium, plus an excess contribution from the beating of the two. These contributions have an exact representation in terms of the $N \times N$ reflection and transmission matrices $r(\omega)$, $t(\omega)$ of the waveguide [7,8],

$$\kappa_{\rm trans}(t) = \int\limits_0^\infty \frac{d\omega}{2\pi} \int\limits_0^\infty \frac{d\omega'}{2\pi} L(\omega-\omega',t)\, f(\omega,T_0) f(\omega',T_0) {\rm Tr}\, T(\omega)T(\omega'), \tag{4.3}$$

$$\kappa_{\rm th}(t) = \int\limits_0^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega'}{2\pi} L(\omega-\omega',t)\, f(\omega,T) f(\omega',T) {\rm Tr}\, Q(\omega)Q(\omega'), \tag{4.4}$$

$$\kappa_{\rm ex}(t) = \int\limits_0^\infty \frac{d\omega}{2\pi} \int\limits_0^\infty \frac{d\omega'}{2\pi} L(\omega-\omega',t)\, 2f(\omega,T_0) f(\omega',T) {\rm Tr}\, T(\omega)Q(\omega'), \tag{4.5}$$

where we have defined

$$L(\omega,t) = \int\limits_0^t dt' \int\limits_0^t dt''\, \exp[i\omega(t'-t'')] = 2\omega^{-2}(1-\cos\omega t), \tag{4.6}$$

$$Q(\omega) = \mathbb{1} - r(\omega)r^\dagger(\omega) - t(\omega)t^\dagger(\omega), \tag{4.7}$$

$$T(\omega) = t(\omega)t^\dagger(\omega). \tag{4.8}$$

Substitution into Eq. (4.2) gives the corresponding contributions to the noise power $P = \bar{I} + P_{\rm trans} + P_{\rm th} + P_{\rm ex}$,

$$P_{\rm trans}(\Omega) = \frac{1}{2}\int\limits_0^\infty \frac{d\omega}{2\pi} f(\omega,T_0) f(\omega+\Omega,T_0) {\rm Tr}\, T(\omega)T(\omega+\Omega) + \{\Omega\to-\Omega\}, \tag{4.9}$$

$$P_{\rm th}(\Omega) = \frac{1}{2}\int\limits_0^\infty \frac{d\omega}{2\pi} f(\omega,T) f(\omega+\Omega,T) {\rm Tr}\, Q(\omega)Q(\omega+\Omega) + \{\Omega\to-\Omega\}, \tag{4.10}$$

$$P_{\rm ex}(\Omega) = \frac{1}{2}\int\limits_0^\infty \frac{d\omega}{2\pi} 2f(\omega,T_0) f(\omega+\Omega,T) {\rm Tr}\, T(\omega)Q(\omega+\Omega) + \{\Omega\to-\Omega\}. \tag{4.11}$$

As in the previous section, we assume a frequency-resolved measurement in an interval $\delta\omega \ll \omega, 1/\tau_{\rm coh}$ with $\Omega \ll \delta\omega$. We may then omit the integral over ω and approximate the argument $\omega \pm \Omega$ in the functions f by ω. We take the ensemble average $\langle\cdots\rangle$ of the noise power, in which case the contributions from $\pm\Omega$ are the same. Finally, we insert the incident current $I_0 = f(\omega,T_0)N\delta\omega/2\pi$, to arrive at

$$P_{\rm trans}(\Omega) = (2\pi/N\delta\omega) I_0^2 \langle N^{-1}{\rm Tr}\, T(\omega)T(\omega+\Omega)\rangle, \tag{4.12}$$

$$P_{\rm th}(\Omega) = (N\delta\omega/2\pi) f^2(\omega,T) \langle N^{-1}{\rm Tr}\, Q(\omega)Q(\omega+\Omega)\rangle, \tag{4.13}$$

$$P_{\rm ex}(\Omega) = 2I_0 f(\omega,T) \langle N^{-1}{\rm Tr}\, T(\omega)Q(\omega+\Omega)\rangle. \tag{4.14}$$

It remains to evaluate the ensemble averages. This is done in the appendix, by extending the approach of Ref. [13] to correlators of reflection and transmission matrices at different frequencies. The calculation applies to the diffusive regime that the length L of the waveguide is large compared to the mean free path l, but still small compared to the localization length Nl. (The absorption length ξ_a is also assumed to be $\gg l$.) The results are

$$\langle N^{-1}\mathrm{Tr}\, T(\omega)T(\omega+\Omega)\rangle = \frac{8D}{c\xi_a}\int_0^s ds'\, K(s',s)\frac{\cosh^2(s-s')}{\sinh^2 s}, \tag{4.15}$$

$$\langle N^{-1}\mathrm{Tr}\, Q(\omega)Q(\omega+\Omega)\rangle = \frac{8D}{c\xi_a}\int_0^s ds'\, K(s',s)\frac{[\cosh s' - \cosh(s-s')]^2}{\sinh^2 s}, \tag{4.16}$$

$$\langle N^{-1}\mathrm{Tr}\, T(\omega)Q(\omega+\Omega)\rangle = \frac{8D}{c\xi_a}\int_0^s ds'\, K(s',s)\frac{\cosh(s-s')\cosh s' - \cosh^2(s-s')}{\sinh^2 s}, \tag{4.17}$$

where $s = L/\xi_a$ and the kernel $K(s', s)$ is defined in Eq. (A29). The combination of Eqs. (4.12)–(4.17) agrees precisely with the results (3.6)–(3.8) of the semiclassical theory. The quantum theory is more general than the semiclassical theory, because it can describe the effects of wave localization. The method of Ref. [13] gives corrections to the above results in a power series in L/Nl. We will not pursue this investigation here.

V. DISCUSSION

We have presented a theory for the frequency dependence of the noise power spectrum $P(\Omega)$ in an absorbing or amplifying disordered waveguide. The frequency dependence is governed by two time scales, the absorption or amplification time τ_a and the diffusion time L^2/D, both of which are assumed to be much greater than the coherence time $\tau_{\rm coh}$. A simplified description is obtained, in the absorbing case, for lengths L much greater than the absorption length $\xi_a = \sqrt{D\tau_a}$, and, in the amplifying case, close to the laser threshold at $L = \pi\xi_a$. We will discuss these two cases separately.

A. Absorbing medium

The general formulas (3.6)–(3.8) for $P = \bar{I} + P_{\rm th} + P_{\rm trans} + P_{\rm ex}$ simplify for $L \gg \xi_a$ to Eqs. (3.10)–(3.12). To characterize the frequency dependence we define the characteristic frequency Ω_c as the frequency at which the super-Poissonian noise has dropped by a factor of two:

$$P(\Omega_c) - \bar{I} = \tfrac{1}{2}\left(P(0) - \bar{I}\right). \tag{5.1}$$

In the absence of any external illumination ($I_0 = 0$) we have, from Eq. (3.10),

$$P = \bar{I}_{\rm th}\left(1 + \frac{f}{1+x}\right), \qquad \bar{I}_{\rm th} = \frac{4Df}{c\xi_a}(N\delta\omega/2\pi), \tag{5.2}$$

with $x = \mathrm{Re}\,\sqrt{1 - i\Omega\tau_a}$, hence $\Omega_c = 17/\tau_a$. If the illumination is in the coherent state from a laser, then we have, from Eq. (3.12),

$$P = \bar{I}_{\rm trans}\left(1 + f\frac{1+2x}{x+x^2}\right), \qquad \bar{I}_{\rm trans} = \frac{8DI_0}{c\xi_a}e^{-s}, \tag{5.3}$$

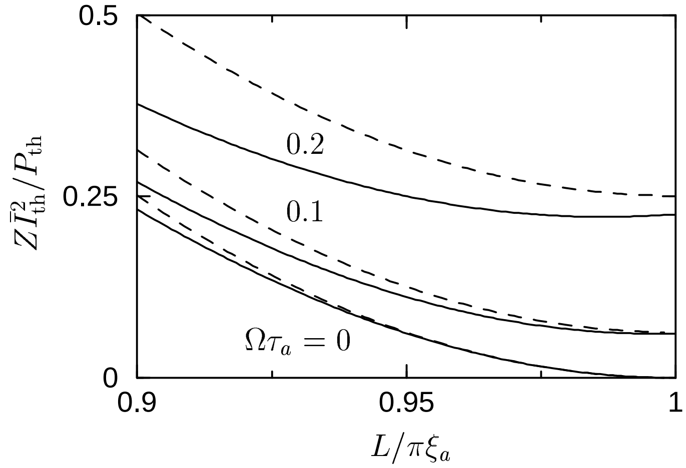

FIG. 6. Ratio of $\bar{I}_{\rm th}^2$ and $P_{\rm th}$ in an amplifying waveguide as a function of its length for different frequencies, computed from Eqs. (3.4) and (3.6). The approximation (5.5) valid near threshold for small frequencies is shown dashed.

here $\Omega_c = 9/\tau_a$. In both these cases the diffusion time does not enter in the frequency dependence. This is different for illumination by a thermal source at temperature T_0 much greater than the temperature of the medium. From Eq. (3.11), with $f_0 = f(\omega, T_0)$, we then have

$$P_{\rm trans}(\Omega) = \bar{I}_{\rm trans}\left(1 + \frac{f_0}{2}e^{-s}\left[\frac{1 - e^{-2s(x-1)}}{x-1} + \frac{3x+2}{x^2+x}\right]\right). \tag{5.4}$$

The characteristic frequency $\Omega_c = (64D/L^2\tau_a^3)^{1/4}$ now contains both the diffusion time and the absorption time.

B. Amplifying medium

In the amplifying case the noise power becomes more and more strongly peaked near zero frequency with increasing amplification. Close to the laser threshold at $s = \pi$ the frequency dependence of $P_{\rm th}$ for small frequencies $\Omega\tau_a \ll 1$ has the form

$$P_{\rm th} = \frac{Z\bar{I}_{\rm th}^2}{2\pi[\Omega^2\tau_a^2 + 4(1 - s/\pi)^2]}, \qquad \bar{I}_{\rm th} = \frac{4f}{Z(\pi - s)}. \tag{5.5}$$

Here again $Z = (c\xi_a/2D)(2\pi/N\delta\omega)$. Close to threshold the peak in the noise power spectrum has a Lorentzian lineshape with half-width $\Omega_c = (2/\tau_a)(1 - L/\pi\xi_a)$. At the laser threshold both $P_{\rm th}$ and $\bar{I}_{\rm th}$ diverge, but the ratio $\bar{I}_{\rm th}^2/P_{\rm th}$ remains finite (see Fig. 6).

Finally, we note the fundamental difference between the time scales appearing in the noise spectrum for photons, on the one hand, and electrons, on the other hand. The absorption or amplification time τ_a obviously has no electronic analogue. The diffusion time L^2/D appears in both contexts, however, the electronic noise spectrum remains frequency independent for $\Omega > D/L^2$ [14]. The reason for the difference is screening of electronic charge. As a result the characteristic frequency scale for electronic current fluctuations is the inverse scattering time D/l^2, which is much greater than the inverse diffusion time D/L^2.

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APPENDIX A: CORRELATORS OF REFLECTION AND TRANSMISSION MATRICES

To compute the noise power spectrum in the quantum mechanical approach of Sec. V, we need the correlators of reflection and transmission matrices $t(\omega_\pm)$ and $r(\omega_\pm)$ at two different frequencies $\omega_\pm = \omega \pm \Omega/2$. (For $\Omega \ll \omega$ this is the same as the correlator at frequencies ω and $\omega + \Omega$.) We calculate these correlators for a waveguide geometry in the diffusive regime, by extending the equal-frequency ($\Omega = 0$) theory of Brouwer [13].

Upon attachment of a short segment of length δL to one end of the waveguide of length L, the transmission and reflection matrices change according to

$$t \to t_{\delta L}(1 + r r_{\delta L})t, \tag{A1a}$$
$$r \to r'_{\delta L} + t_{\delta L}(1 + r r_{\delta L}) r t^{\mathrm{T}}_{\delta L}, \tag{A1b}$$

where the superscript T indicates the transpose of a matrix. (Because of reciprocity the transmission matrix from left to right equals the transpose of the transmission matrix from right to left.) The transmission matrix $t_{\delta L}$ of the short segment at frequency $\omega_\pm$ may be chosen proportional to the unit matrix,

$$t_{\delta L} = \left(1 - \frac{\delta L}{2l'} - \frac{\delta L}{2c'\tau_a} \pm \frac{i\Omega\delta L}{2c'}\right)\mathbb{1}. \tag{A2}$$

The mean free path $l' = 4l/3$ and the velocity $c' = c/2$ represent a weighted average over the N transverse modes in the waveguide.

Unitarity of the scattering matrix dictates that the reflection matrix from the left of the short segment is related to the reflection matrix from the right by $r'_{\delta L} = -r^\dagger_{\delta L}$. We abbreviate $r_{\delta L} \equiv \delta r$. The matrix δr is symmetric (because of reciprocity), with zero mean and variance

$$\langle \delta r_{kl} \delta r^*_{mn} \rangle = (N+1)^{-1}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\delta L/l'. \tag{A3}$$

The resulting change in the matrix products $tt^\dagger$ and $rr^\dagger$ is

$$\begin{aligned} tt^\dagger \to{} & (1 - \delta L/l' - \delta L/c'\tau_a)tt^\dagger + (r\delta r t)(r\delta r t)^\dagger \\ & + r\delta r t t^\dagger + (r\delta r t t^\dagger)^\dagger, \end{aligned} \tag{A4a}$$
$$\begin{aligned} rr^\dagger \to{} & (1 - 2\delta L/l' - 2\delta L/c'\tau_a)rr^\dagger + (r\delta r r)(r\delta r r)^\dagger + \delta r^\dagger \delta r \\ & + r\delta r r r^\dagger + (r\delta r r r^\dagger)^\dagger - r\delta r - (r\delta r)^\dagger. \end{aligned} \tag{A4b}$$

The frequency Ω does not appear explicitly in these increments.

We define the following ensemble averages

$$\mathcal{R} = \langle N^{-1}\mathrm{Tr}\,(\mathbb{1} - rr^\dagger)\rangle, \tag{A5}$$
$$\mathcal{C} = \langle N^{-1}\mathrm{Tr}\,(\mathbb{1} - r_-r_+^\dagger)\rangle, \tag{A6}$$
$$\mathcal{T} = \langle N^{-1}\mathrm{Tr}\,tt^\dagger\rangle, \tag{A7}$$

where r, t are evaluated at frequency ω and $r_\pm, t_\pm$ at frequency $\omega \pm \Omega/2$. Similarly, we define the correlators

$$C_{rr} = \langle N^{-1}\mathrm{Tr}\,(\mathbb{1} - r_-r_-^\dagger)(\mathbb{1} - r_+r_+^\dagger)\rangle, \tag{A8}$$
$$C_{rt} = \langle N^{-1}\mathrm{Tr}\,(\mathbb{1} - r_-r_-^\dagger)t_+t_+^\dagger\rangle, \tag{A9}$$
$$C_{tt} = \langle N^{-1}\mathrm{Tr}\,t_-t_-^\dagger t_+t_+^\dagger\rangle. \tag{A10}$$

We will see that, in the diffusive regime, these 6 quantities satisfy a coupled set of ordinary differential equations in L.

The diffusive regime corresponds to the large-N limit, in which the length L of the waveguide is much less than the localization length Nl. In this limit we may replace Eq. (A3) by $\langle \delta r_{kl}\delta r^*_{mn}\rangle = (\delta L/Nl')\delta_{km}\delta_{ln}$. In the large-$N$ limit we may also replace averages of products of traces by products of averages of traces. From Eq. (A4) we thus obtain the diffential equations

$$l'\frac{d\mathcal{R}}{dL} = 2\gamma(1-\mathcal{R}) - \mathcal{R}^2, \tag{A11}$$
$$l'\frac{d\mathcal{C}}{dL} = 2\gamma(1+i\Omega\tau_a)(1-\mathcal{C}) - \mathcal{C}^2, \tag{A12}$$
$$l'\frac{d\mathcal{T}}{dL} = -\gamma\mathcal{T} - \mathcal{R}\mathcal{T}, \tag{A13}$$
$$l'\frac{dC_{rr}}{dL} = -(4\gamma + \mathcal{C} + \mathcal{C}^* + 2\mathcal{R})C_{rr} + 2\mathcal{R}(\mathcal{R}+2\gamma), \tag{A14}$$
$$l'\frac{dC_{rt}}{dL} = -(3\gamma + \mathcal{C} + \mathcal{C}^* + \mathcal{R})C_{rt} - \mathcal{T}C_{rr} + 2(\mathcal{R}+\gamma)\mathcal{T}, \tag{A15}$$
$$l'\frac{dC_{tt}}{dL} = -(2\gamma + \mathcal{C} + \mathcal{C}^*)C_{tt} - 2\mathcal{T}C_{rt} + 2\mathcal{T}^2, \tag{A16}$$

with the definition $\gamma = l'/c'\tau_a$. The initial conditions are that each of these 6 quantities $\to 1$ for $L \to 0$.

This set of differential equations may be simplified further if we assume, as we did in the semiclassical theory, that the mean free path is small compared to both the absorption length and the length of the waveguide. All 6 quantities (A5)–(A10) are of order $\sqrt{\gamma}$, which is $\ll 1$ if $l' \ll c'\tau_a$, so that we obtain in leading order

$$l'\frac{d\mathcal{R}}{dL} = 2\gamma - \mathcal{R}^2, \tag{A17}$$
$$l'\frac{d\mathcal{C}}{dL} = 2\gamma(1+i\Omega\tau_a) - \mathcal{C}^2, \tag{A18}$$

$$l' \frac{d\mathcal{T}}{dL} = -\mathcal{R}\mathcal{T}, \tag{A19}$$

$$l' \frac{dC_{rr}}{dL} = -(\mathcal{C} + \mathcal{C}^* + 2\mathcal{R})C_{rr} + 2\mathcal{R}^2, \tag{A20}$$

$$l' \frac{dC_{rt}}{dL} = -(\mathcal{C} + \mathcal{C}^* + \mathcal{R})C_{rt} - \mathcal{T}C_{rr} + 2\mathcal{R}\mathcal{T}, \tag{A21}$$

$$l' \frac{dC_{tt}}{dL} = -(\mathcal{C} + \mathcal{C}^*)C_{tt} - 2\mathcal{T}C_{rt} + 2\mathcal{T}^2. \tag{A22}$$

As initial condition we should now take that the product of each quantity with L remains finite when $L \to 0$.

Although the differential equations are coupled, they may be solved separately for $\mathcal{R}$, $\mathcal{C}$, $\mathcal{T}$, C_{rr}, C_{rt}, C_{tt}, in that order. In terms of the rescaled length $s = (2\gamma)^{1/2}L/l' = L/\xi_a$, the results are

$$\mathcal{R} = \frac{(2\gamma)^{1/2}}{\tanh s}, \tag{A23}$$

$$\mathcal{C} = \frac{(2\gamma)^{1/2}\sqrt{1 + i\Omega\tau_a}}{\tanh s\sqrt{1 + i\Omega\tau_a}}, \tag{A24}$$

$$\mathcal{T} = \frac{(2\gamma)^{1/2}}{\sinh s}, \tag{A25}$$

$$C_{rr} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds'\, K(s', s) \cosh^2 s', \tag{A26}$$

$$C_{rt} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds'\, K(s', s) \cosh(s - s') \cosh s', \tag{A27}$$

$$C_{tt} = \frac{(8\gamma)^{1/2}}{\sinh^2 s} \int_0^s ds'\, K(s', s) \cosh^2(s - s'), \tag{A28}$$

where the kernel K is defined by

$$K(s', s) = \left|\sinh s'\sqrt{1 + i\Omega\tau_a}\right|^2 \left|\sinh s\sqrt{1 + i\Omega\tau_a}\right|^{-2}. \tag{A29}$$

These are the expressions used in Sec. 4 (where we have also substituted $\sqrt{2\gamma} = 4D/c\xi_a$). The remaining integrals over s' may be done analytically, but the resulting expressions are rather lengthy so we do not record them here. For $\Omega = 0$ our results reduce to those of Brouwer [13] (up to a misprint in Eq. (13c) of that paper, where the plus and minus signs in the expression between brackets should be interchanged).

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